# **Stochastic Framework for Symmetric Affine Matching between Point Sets**

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#### Abstract

This paper presents a new approach to obtain symmetry in point matching problem. Here, symmetric matching means the essential property that the choices of source and target should not determine the eventual matching results. Most earlier approaches to achieve symmetric matching have been in deterministic fashions, where symmetry constraints are added into the matching cost functions to impose source-target symmetric property during the matching process. Nevertheless, these modified cost functions cannot generally converge to real ground truth, and further, the perfect source-target symmetry cannot be achieved. Given initial forward and backward matching matrices pair, computed from any reasonable matching strategies, our approach yields perfectly symmetric mapping matrices from a stochastic framework that simultaneously considers the errors underneath the initial matching matrices and the imperfectness of the symmetry constraint. An iterative generalized total least square (GTLS) strategy has been developed such that perfect source-target symmetry is imposed.

## 1. Introduction

Intuitively, the matching problem should be symmetric in nature if the matching pair is in continuous domain, i.e., the ground truth correspondences established between the two data inputs should be independent of their order to the matching process. However, the source-target symmetry cannot be achieved even in matching very simple object. One of the most common point matching algorithm, Iterative Closet Point (ICP) [1] is used to demonstrate this fact (Fig.1). Conventional matching algorithms adopt the term source (S) for the input data being transformed in the matching process, while the one keeping fixed for the source to match is referred to as target (T). In this paper,  $T_{12}$  and  $T_{21}$  are referred to the transformations solved by the forward and backward matching processes respectively.

Such asymmetric nature of the conventional approaches

raises a question. If there are two different transformation results, which one should we take? Should we pick one and perform a simple inverse? It would not be an easy question as in real matching problem the ground truth is always unavailable. To solve this problem we must have a matching cost function which does not depend on the order of S and T, i.e., E(S,T) = E(T,S) [5]. The following matching cost function is possible to achieve source-target symmetry:

$$E(S,T) = w \times E_{Sim}(S,T) + (1-w) \times E_{Sim}(T,S)$$
(1)

where  $E_{Sim}$  measures the similarity (e.g., image intensity and geometrical properties) between the data sets. The simplest case for the weighting factor w is 0.5.

Christensen [2] enforced an inverse consistency constraint with the similarity metric during the optimization process. In [2], the symmetric property in Eq. (1) was achieved, however, by separating the forward and reverse registration processes into two phases with an additional inverse consistency constraint:

$$E_f(S,T) = E_{Sim}(S,T) + \rho \times E_{Cons}(S,T)$$
(2)

 $E_r(T,S) = E_{Sim}(T,S) + \rho \times E_{Cons}(T,S)$  (3)  $E_{Cons}(S,T)$  and  $E_{Cons}(T,S)$  are equivalent to  $||T_{12} - T_{21}^{-1}||$  and  $||T_{21} - T_{12}^{-1}||$  respectively. Eq. (2) and (3) are alternately optimized until convergence. The main problem of Eq. (2) and (3) is that the consistency property is only part of the overall cost function. This formulation is only

All the above symmetric formulations yield a 1-to-1 symmetric mapping that is *deterministic* in nature and assuming both the observation (matching results) and the model (symmetric relation) are not perturbed by any errors:

*asymptotically* inverse consistent when  $\rho$  tends to infinity.

$$T_{12} * T_{21} = I \tag{4}$$

However, unless you have a continuous representation of the object and also a precise transformation model, otherwise it would not be true in general. These imperfect transformation matrices mean that their underlying stochastic uncertainties should not be omitted.





Figure 1. Forward and backward matching of 2 point sets to demonstrate inconsistent correspondences due to switch of the input order. Number of iterations in this ICP example is 149.

We propose a stochastic framework for matching problems which generates perfect source-target symmetric mapping between the data sets. Instead of imposing sourcetarget symmetry in a *deterministic* and *asymptotic* sense, we enforce the symmetric property with the systematic considerations of the stochastic uncertainties of the input forward, backward transformation matrices and the symmetry constraint<sup>1</sup> to achieve *perfect* source-target symmetry. Given the forward and backward matching results, an iterative fitting process is performed until a set of new forward and backward transformation matrices are obtained which are perfectly inverse to each other. The fitting process is solved with the adoption of Generalized Total Least Square(GTLS) technique [6] which allows simultaneous considerations of all the errors in the input transformation matrices and the symmetry constraint. Here, we apply our stochastic framework to point matching problems and study the robustness of our system towards different amounts and types of noise on the input point sets. The framework can be applied to volumetric image registration problems with minor modification [7].

# 2. Stochastic framework for symmetric matching

# 2.1. Stochastic symmetry constraint

As we have stated above, the forward and backward transformation matrices obtained from any matching algorithm are going to be error-perturbed. Simply combine them deterministically may not be a good way to utilize the information from the forward and backward process. In this paper, we suggest that one should model the source-target symmetry stochastically with the simultaneous consideration of the underlying stochastic uncertainties within the forward and backward transformation matrices and hence the imperfectness of the symmetry constraint, i.e,

$$(T_{12}^G + E_{T_{12}}) * (T_{21}^G + E_{T_{21}}) = I + R$$
(5)

With Eq.(5), the observation and the model are no longer perfect but perturbed by noise.  $E_{T_{12}}$  and  $E_{T_{21}}$  are the errors associated with the ground truth transformations  $T_{12}^G$  and  $T_{21}^G$ . *R* is the error over identity due to 2 asymmetric matrices<sup>2</sup>. In our current error model, we assume all the elements in the error matrices are zero mean and are independent to each other.

# 2.2. Symmetry enforcement through iterative GTLS fitting

Since both the observation and the model are perturbed by noise, we adopted the Total Least Square (TLS) approach instead of the general Least Square (LS) to solve our problem. Also, as the stochastic property is not the same for every entry and some of the entries are error free, in order to solve the problem while considering all the errors simultaneously, a Generalized Total Least Square (GTLS) [6] approach is adopted.

The initial input of our framework are the transposed and permuted version of  $T_{12}$  and  $T_{21}$  (transformation matrices from the forward and backward matching process):

$$Q_{12}^{(0)} = T_{12}^T * P \qquad \qquad Q_{21}^{(0)} = T_{21}^T * P \qquad (6)$$

$$invQ_{12}^{(0)} = (T_{12}^{-1})^T * P$$
  $invQ_{21}^{(0)} = (T_{21}^{-1})^T * P$  (7)

where 
$$P = P_{14} * P_{24} * P_{34}$$
 and  
 $P_{14} = (e_4 e_2 e_3 e_1), P_{24} = (e_1 e_4 e_3 e_2), P_{34} = (e_1 e_2 e_4 e_3)$   
 $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (8)

<sup>2</sup>We simply assume all the entries in R have the same stochastic uncertainty and set it as  $\Delta_r$ .



<sup>&</sup>lt;sup>1</sup>There are works about stochastic ICP, e.g., in [4] which impose the stochastic property on the point's location while we impose the stochastic property on the transformation matrices and also the symmetry constraint.



Figure 2. First row: different source point sets to the whole matching process. Second row: corresponding target point sets with different combinations of deformation and noise. Third row: the visual results of the forward process are shown. Different colors represent the source points warped by different transformations, red triangles( $T_{12}$ ), green stars( $T_{12}^*$ ), blue triangles( $T_{21}^{-1}$ ).

It is done because the first column of  $Q_{12}^{(0)}$  and  $Q_{21}^{(0)}$  will become error free. Our objective is to utilize the GTLS formulation to fit the transformation matrices iteratively, considering the errors in the transformation matrices and the sourcetarget symmetry constraint until the forward and backward directions are symmetric:

$$\begin{bmatrix} Q_{12}^{(i)} \\ invQ_{21}^{(i)} \end{bmatrix} X \approx \begin{bmatrix} I \\ I \end{bmatrix} \begin{bmatrix} invQ_{12}^{(i)} \\ Q_{21}^{(i)} \end{bmatrix} Y \approx \begin{bmatrix} I \\ I \end{bmatrix}$$
(9)

with the corresponding stochastic property in the noise data:

$$\begin{bmatrix} E_{Q_{12}}^{(i)} & R \\ E_{invQ_{21}}^{(i)} & R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_{invQ_{12}}^{(i)} & R \\ E_{Q_{21}}^{(i)} & R \end{bmatrix} \quad (10)$$

where i is the iteration number. And

$$X^{-1} = invQ_{21}^{(i+1)} \text{ and } P * X * P = Q_{21}^{(i+1)}$$
(11)

$$Y^{-1} = invQ_{12}^{(i+1)}$$
 and  $P * Y * P = Q_{12}^{(i+1)}$  (12)

The process iterates until the degree of asymmetry  $(\epsilon)$  is negligible:

$$\epsilon = ||(P * X)^T * (P * Y)^T - I||_F < \text{threshold}$$
(13)

and the GTLS solution matrices will be  $T_{21}^* = (P * X)^T$ ,  $T_{12}^* = (P * Y)^T$ . The error matrices  $E_{Q_{12}}$ ,  $E_{invQ_{12}}$  for  $Q_{12}$ ,  $invQ_{12}$  are:

$$E_{Q_{12}}^{(i)} = |Q_{12}^{(i)} - invQ_{21}^{(i)}| \quad E_{invQ_{12}}^{(i)} = |Q_{21}^{(i)} - invQ_{12}^{(i)}|$$
 (14)

i.e., absolute difference between 2 matrices which can be treated as the error's upper bound. And the first column of  $E_{Q_{12}}$  is dropped as the first column of  $Q_{12}$  is error free. The error matrices  $E_{invQ_{21}}$ ,  $E_{Q_{21}}$  are formed respectively by:

$$E_{invQ_{21}}^{(i)} = \frac{(1-\alpha)}{\alpha} * E_{Q_{12}}^{(i)} \quad E_{Q_{21}}^{(i)} = \frac{(1-\alpha)}{\alpha} * E_{invQ_{12}}^{(i)}$$
(15)

where  $\alpha$  is the weighting on the error of the forward matching process and can be picked by any prior knowledge of the input data or matching process. Therefore,  $\alpha$  mimics weighting factor w in Eq.(1) in a stochastic sense. To simplify the model  $\alpha$  is set to 0.5 in this paper. The error matrix R for the source-target symmetry constraint is fixed as the initial input stochastic consistent model is kept unchanged.

The error equilibration matrices for solving X in Eq.(9) are obtained from the Cholesky decomposition of the error covariance matrices C and D, where  $C = \Delta^T \Delta$ ,  $D = \Delta \Delta^T$ ,  $\Delta = \begin{bmatrix} E_{Q_{12}} & R \\ E_{invQ_{21}} & R \end{bmatrix}$ . Similarly Y is solved.

## 3. Experiments and conclusion

We have applied our stochastic symmetric model on different point sets. There are four different point sets in Fig.2. Two transformations, one is smaller deformed and one is larger deformed, are applied on each of the point set to form



the corresponding target point set. Then, different amounts of gaussian noise and impulse noise are separately added to both the source and target point sets in the experiment. The forward and backward transformation matrices for our system inputs are obtained by the Robust point matching algorithm RPM [3]. Results evaluation is based on the sum of squared distance (SSD) between the points in the warped source point set and the target point set. Fig.2 shows examples with different degree of deformation and combination of noise. The corresponding SSD is shown in Table.1, the order of SSD is  $10^4$  and  $10^6$  for the gaussian and impulse noise respectively. Notice that the error should be compared within the same direction of the transformation, i.e.,  $(T_{12},$ GTLS  $T_{12}, T_{21}^{-1}$ ) in the forward direction,  $(T_{21}, \text{GTLS } T_{21},$  $T_{12}^{-1}$ ) in the backward direction. The SSD of GTLS  $T_{12}$  is colored in green if it outperforms both  $T_{12}$  and  $T_{21}^{-1}$ , similarly for GTLS  $T_{21}$ . Notice that as shown in the table, even  $T_{12}$  gives a better result in the forward direction,  $T_{12}^{-1}$  may indeed give you a much worse result in the backward direction. Our GTLS approach yields a unique mapping with reasonable results in both directions.

We presented a novel framework for modelling the source-target symmetry stochastically, by simultaneously considering the stochastic uncertainties on both of the transformation matrices and the symmetry constraint through the Generalized Total Least square fitting from the transformation matrices obtained after the matching process. With our stochastic consistency model, symmetry property can be imposed perfectly with the consideration of any other similarity constraints. This work is supported in part by Hong Kong Research Grants Council CERG-HKUST6252/04E and by China 973 Program (2003CB716103).

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			Posi	tion Errors	(SSD) - Fi	ish				
		Gauc	rian Noica	(S D)	Small Def	ormation	Impulsa	Noica (Pro	nortion)	
Transformation	1	2	3	(3.D)	5	0.1	0.2	0.3	0.4	0.5
T <sub>12</sub>	143	100	130	270	173	24	26	51	61	91
GTLS T <sub>12</sub>	145	99	125	283	175	29	32	61	74	98
$T_{21}^{-1}$	153	107	135	330	206	48	46	77	95	110
	216	142	187	408	228	30	31	51	69	94
$T_{21}^{-1}$	220	140	197	419	218	28	34	55	72	104
T <sub>12</sub>	260	1/1	244	520	260 Larga Daf	35 ormation	48	68	79	126
		Gaussian Noise(S.D)			Large Der	Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T <sub>12</sub>	178	218	241	285	538	39	45	52	63	123
GTLS T <sub>12</sub>	187	233	250	285	615	39	48	55	67	104
T <sub>21</sub>	197	252	277	346	728	40	56	86	71	102
	94	124	144	168	366	210	271	328	441	583
$T^{-1}$	91	119	133	133	264	217	290	570	40.5 510	756
<sup>1</sup> 12	91	122	144	178	304	237	334	0.54	519	/30
Position Errors (SSD) - Smile face										
		Gaus	sian Noise	(S.D)	Smail Del	amation	Impulse I	Noise (Pro	portion)	
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T <sub>12</sub>	11	41	101	164	259	56	148	205	268	373
GTLS T12	10	41	89	161	271	89	169	224	292	405
$T_{21}^{-1}$	12	46	96	177	331	197	201	250	319	434
T <sub>21</sub>	14	57	114	213	375	86	165	228	252	382
1	13	51	110	201	327	72	1//	247	265	401
T <sub>12</sub>	15	53	134	217	346	81	205	286	291	436
	Large Deformation							Noise (Pro	nortion)	
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T <sub>12</sub>	15	65	153	230	436	121	271	266	389	504
GTLS T <sub>12</sub>	16	60	134	236	479	112	231	269	411	554
$T_{21}^{-1}$	22	70	141	283	597	129	249	320	481	647
	12	42	91	165	333	77	150	204	301	393
$T^{-1}$	9	37	0/	147	292	127	208	231	459	434
1 10	10	41	101	150	511	15/	447	200	4.30	202
12										
			Positi	on Errors (	SSD) - Ra	ptor				1
		Gaus	Positi sian Noise	on Errors (	SSD) - Ra Small Def	ptor ormation	Impulse 1	Noise (Pro	portion)	1
Transformation	1	Gaus 2	Positi sian Noise 3	on Errors ( (S.D) 4	(SSD) - Raj Small Def	ptor ormation 0.1	Impulse 1	Noise (Pro	portion) 0.4	0.5
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$\begin{array}{c} 12 \\ \hline \\ \hline \\ Transformation \\ \hline T_{12} \\ \hline \\ T_{21} \\ \hline \\ T_{21} \\ \hline \\ T_{21} \\ \hline \\ T_{21} \\ \hline \\ T_{12} \\ \hline \\ \hline \\ \\ \\ \hline \\ \\ \\ \\ \hline \\ \\ \\ \\ \\ \\ \hline \\$	1 42 465 50 67 66 65 1 194 192 191 97 99 103 103	Gaus 2 44 48 51 69 66 65 7 88 83 83 95 96 99 99 7 7 6 8 8 11 10 13 13 2 2 53 53 54	Positi sian Noise 3 57 62 83 80 81 81 81 81 214 112 113 117 216 112 113 117 Positi 15 19 19 19 27 3 3 80 77 77 77	(S.D) (S	SSD) - Ra Small Def 72 81 109 115 100 Large Def 265 265 147 147 147 147 147 147 147 147 147 147	ormation           0.1         123           134         144           144         145           ormation         0.1           20         20           20         21           14         13           13         14           13         16           ormation         0.1           20         20           20         20           20         13           14         13           16         66           40         42           0.1         58           61         66           61         66	Impulse 1:           0.2           333           407           379           401           Impulse 1:           0.2           38           37           26           28           36           Impulse 2:           0.2           86           97           111           96           98           107           Impulse 1:           0.2	Noise (Prrc 362) 563 564 587 631 631 810 63 65 62 63 631 81 81 81 81 81 81 81 81 81 81 81 81 81	portion) 0.4 584 648 647 756 756 97 100 0.4 88 97 100 0.4 88 97 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 0.4 100 100 100 100 100 100 100 10	0.5 811 987 995 986 0.5 114 122 130 89 95 106 0.5 236 236 236 249 249 266 0.5 237 280 0.5
$\begin{array}{c} 12 \\ \hline \\ $	1 42 465 50 67 66 65 1 194 192 191 97 99 103 103 1 1 4 5 4 4 7 7 1 4 5 4 9 7 95 4 95 4 95 4 95 4	Gauss 2 44 48 51 69 66 65 7 88 183 95 183 95 96 99 99 7 7 8 8 11 10 13 13 6 aus 2 53 54 57 268	Positi sian Noise 3 57 62 83 80 80 81 81 216 112 216 112 216 112 214 216 112 113 117 Positi 117 Positi 119 19 27 27 sian Noise 3 67 70 77 70 77 70 77 70 77	(S.D) 4 51 53 57 73 57 71 75 (S.D) 4 214 208 211 110 208 211 112 122 122 122 122 123 33 30 33 (S.D) 4 57 58 57 57 57 57 57 57 57 57 57 57	SSD) - Ra Small Def 5 72 81 92 115 108 110 Large Def 25 251 266 283 155 245 266 283 155 147 143 (SSD) - Br Small Def 5 99 94 103 124 122 132 Large Def 5 99 104 115 5 99	ormation           0.1           123           134           144           144           144           144           145           ormation           0.1           20           21           14           ain           ormation           0.1           20           21           14           ain           ormation           0.1           661           660           40           42           46           ormation           0.1           58           61           66           40	Impulse 1           0.2           333           407           474           379           401           451           Impulse 1           0.2           38           37           42           26           0.2           38           36           100           97           111           96           98           1007           Impulse 1           99           96           9111           70	Noise (Proc 0.3 521 556 605 564 631 Noise (Proc 0.3 65 62 69 44 52 81 127 134 142 156 Voise (Proc 0.3 142 157 134 141 156	portion) 0.4 584 713 647 756 576 577 110 68 97 110 68 97 110 68 81 107 281 107 108 108 109 109 109 109 109 109 109 109	0.5 811 957 986 987 986 986 986 986 986 986 986 986 95 106 0.5 236 236 236 236 236 236 232 240 249 266 0.5 237 237 2380 313 244
$\begin{array}{c} 12 \\ \hline \\ $	1 42 465 50 67 66 65 1 194 192 191 192 191 97 99 99 103 103 103 1 4 5 4 4 7 7 1 45 49 54 232	Gauss 2 44 51 69 66 65 7 88 5 88 5 96 6 5 88 5 96 6 88 99 99 99 99 99 99 7 8 8 83 99 99 99 99 99 99 99 10 11 10 10 10 10 10 10 10 10 10 10 10	Positi sian Noise 3 54 80 81 81 81 81 81 81 81 81 81 81 215 214 214 216 112 214 216 112 113 81 81 81 81 81 81 81 81 81 81 81 81 81	(S.D) 4 51 53 57 73 77 75 (S.D) 4 214 214 208 208 208 208 208 208 208 208	SSD) - Ra Small Def 72 72 115 108 110 Large Def 251 266 283 155 143 (SSD) - Br Small Def 5 99 4 103 124 122 132 Large Def 5 99 104 115 592	ormation           0.1           123           134           144           144           144           144           144           145           ormation           0.1           20           20           21           14           13           14           13           14           13           14           31           14           31           61           66           40           0.1           57           61           66           40           0.1           58           61           66           40           42	Impulse J           0.2           333           407           474           379           401           Impulse J           0.2           38           0.2           38           0.2           38           0.2           38           0.2           86           96           1111           70           96           1111	Noise (Proc 0.3 521 564 587 663 663 663 663 663 663 663 664 663 664 663 664 663 664 663 664 664	portion) 0.4 584 713 647 756 5756 597 110 68 97 110 68 97 110 68 97 110 68 81 81 81 187 205 185 225 243 168 169 169 169 169 169 169 169 169	0.5 811 905 887 996 986 986 986 986 986 986 986 986 986
$\begin{array}{c} 12 \\ \hline \\ \hline \\ Transformation \\ \hline T_{12} \\ \hline \\ T_{21} \\ \hline \\ T_{12} \\ \hline \\ \hline \\ \hline \\ T_{12} \\ \hline \\ \hline \\ \hline \\ T_{12} \\ \hline \\ \hline \\ \hline \\ \hline \\ T_{12} \\ \hline \\ \hline \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \\ \\ \hline \\ \\ \hline \hline \\ \hline \\ \hline \hline \\ \hline \\ \hline \\ \hline \hline \\ \hline \\ \hline \hline \hline \hline \hline \hline \hline \hline \\ \hline \hline$	1 42 465 50 67 66 65 1 194 194 199 191 97 99 90 103 103 103	Gauss 2 44 48 51 69 66 5 2 183 183 95 96 6 5 8 8 183 95 96 99 99 99 99 99 7 7 8 11 10 13 2 95 54 55 7 268 54 838	Positi sian Noise 3 57 62 83 80 81 3 215 214 216 112 216 112 113 117 Positi sian Noise 3 18 14 15 19 19 19 27 27 sian Noise 3 67 70 77 70 77 70 27 224	(S.D) 4 51 53 57 73 75 (S.D) 4 214 208 4 214 214 214 211 110 112 213 30 33 30 (S.D) 4 58 64 320 363 363 363 363 363 363 363 36	SSD) - Ra Small Def 72 72 115 108 110 Large Def 251 251 263 155 147 (SSD) - Br Small Def 5 99 103 124 123 124 123 124 123 124 132 Large Def 5 99 94 103 124 132 Large Def 5 99 94 103 124 132 Large Def 5 99 94 103 124 125 132 132 132 132 132 133 135 135 135 135 135 135 135 135 135	ormation           0.1           123           134           144           144           144           144           144           144           144           144           144           144           144           147           156           ormation           20           20           20           21           14           13           14           13           6           40           42           46           ormation	Impulse 12 0.2 333 407 474 474 379 401 474 451 451 451 452 86 97 1111 102 98 107 102 99 96 1111 70 86 131	Noise (Proc 0.3 565 564 631 Noise (Proc 0.3 65 62 69 44 452 881 81 142 142 142 142 142 157 154 156 Noise (Proc 0.3 117 156 156 156 160 157 148 1157 156 156 156 156 156 156 156 156 156 156	portion) 0.4 584 648 713 647 756 756 756 756 88 97 10 68 87 205 185 194 10 0.4 187 205 185 221 225 229	0.5 811 905 9861 9986 986 986 986 986 986 986 986 986 98

Table 1. Sum of squared distances (SSD) of different data sets under different combinations of transformation, deformation and noise. Gaussian noise with Standard.Deviation = 1,2,3,4,5 and Impulse noise with proportion = 0.1,0.2,0.3,0.4,0.5 to the number of original points. Transformations for forward direction: ( $T_{12}$ , GTLS  $T_{12}$ ,  $T_{21}^{-1}$ ), backward direction: ( $T_{21}$ , GTLS  $T_{21}$ ,  $T_{12}^{-1}$ ). This table is color-coded.

